

# EXISTENCE OF MULTIPLE SOLUTIONS FOR A QUASILINEAR ELLIPTIC PROBLEM

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## Abstract

In this paper we prove the existence of multiple solutions for a quasilinear elliptic boundary value problem, when the  $p$ -derivative at zero and the  $p$ -derivative at infinity of the nonlinearity are greater than the first eigenvalue of the  $p$ -Laplace operator. Our proof uses bifurcation from infinity and bifurcation from zero to prove the existence of unbounded branches of positive solutions (resp. of negative solutions). We show the existence of multiple solutions and we provide qualitative properties of these solutions.

**Key Words and phrases:** quasilinear elliptic equations, bifurcation theory, multiplicity of solutions.

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## 1 Introduction

In this paper we study the existence of multiple solutions for the quasilinear elliptic boundary value problem

$$\begin{cases} \Delta_p u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , is a bounded and smooth domain,  $1 < p < 2$ ,  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator, and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a nonlinear function such that  $f(0) = 0$  and

$$(f_1) \quad |f(t) - f(s)| \leq C_f |t - s|^{p-1}, \quad \forall s, t \in \mathbb{R},$$

$$(f_2) \quad f'_p(0) := \lim_{t \rightarrow 0} \frac{f(t)}{|t|^{p-2}t} > \lambda_1(p),$$

$$(f_3) \quad f'_p(\infty) := \lim_{t \rightarrow \infty} \frac{f(t)}{|t|^{p-2}t} > \lambda_1(p),$$

$$(f_4) \quad \text{there exists a positive number } \alpha \text{ such that } f(\alpha) \leq 0 \leq f(-\alpha),$$

where  $C_f := \sup_{s \neq t} |f(s) - f(t)|/|s - t|^{p-1} \in \mathbb{R}$ , and  $\lambda_1(p)$  denotes the first eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

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We call  $f'_p(0)$  the  $p$ -derivative at zero and  $f'_p(\infty)$  the  $p$ -derivative at infinity.

We prove that problem (1) has at least four nontrivial solutions, two of them are positive and the other two are negative. We also found some upper and lower bounds for the  $L^\infty$ - norm of these solutions.

**Theorem A.** *If  $f$  satisfies  $(f_1)$ ,  $(f_2)$ ,  $(f_3)$ , and  $(f_4)$  then problem (1) has at least four nontrivial solutions  $u_1, u_2, v_1$ , and  $v_2$ . Moreover, solutions  $u_1$  and  $u_2$  are positive on  $\Omega$ , and solutions  $v_1$  and  $v_2$  are negative on  $\Omega$ . In addition,*

$$\|u_2\|_{L^\infty} < \alpha < \|u_1\|_{L^\infty}$$

and

$$\|v_2\|_{L^\infty} < \alpha < \|v_1\|_{L^\infty}.$$

**Remark:** Actually, the argument we present below allows to prove a more general result: if  $f$  satisfies  $(f_1)$ ,  $(f_2)$ ,  $(f_3)$ , and

$(f'_4)$  there exist numbers  $\alpha > 0$  and  $\tilde{\alpha} < 0$  such that  $f(\alpha) \leq 0 \leq f(\tilde{\alpha})$ ,

then problem (1) has at least four nontrivial solutions  $u_1, u_2, v_1$ , and  $v_2$ . Moreover, solutions  $u_1$  and  $u_2$  are positive on  $\Omega$ , and solutions  $v_1$  and  $v_2$  are negative on  $\Omega$ . In addition,

$$\|u_2\|_{L^\infty} < \alpha < \|u_1\|_{L^\infty}$$

and

$$\|v_2\|_{L^\infty} < |\tilde{\alpha}| < \|v_1\|_{L^\infty}.$$

For the sake of simplicity, from now on we assume hypothesis  $(f_4)$  instead of  $(f'_4)$  (i.e.  $\tilde{\alpha} = -\alpha$ ).

Our proof of Theorem A uses bifurcation from infinity and bifurcation from zero, applied to the problem

$$\begin{cases} \Delta_p u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where  $\lambda > 0$ .

Theorem A is an extension to quasilinear equations of a result due to J. Cossio, S. Herrón, and C. Vélez (see [CHV1]) for the semilinear case. A key ingredient to extend the semilinear result to our situation is to prove that for problem (3) there exist unbounded branches of positive solutions (resp. of negative solutions) emanating from the bifurcation points  $(\infty, \lambda_1/f'_p(\infty))$  and  $(0, \lambda_1/f'_p(\infty))$  (see Theorem 3.1 and Theorem 3.2 in Section 3 below). Theorem 3.1 is very much inspired by a corresponding result in the semilinear case due to Ambrosetti and Hess (see [AH] and Section 4.4 in [AM]), and by Theorem 4.1 in [AGP]. Although our proof of Theorem 3.1 follows the ideas from [AH], [AM] and [AGP], our arguments have several differences with respect to these references, as will be better explained in Section 3. Theorem 3.2, on the other hand, essentially comes from the ideas by Del Pino and Manásevich in [DM].

The existence of solutions to quasilinear elliptic problems like (3) has been widely investigated. Let us mention, besides [AGP] and [DM], papers [DGTU], [Drab] and [DQ], the books [FNSS] and [DKT], and the references therein. A. Ambrosetti et al. in [AGP] showed the existence of an unbounded branch of positive solutions of problem (3) emanating from either zero or infinity when  $f(u) \simeq u^{p-1}$  near 0 or near infinity;

they used a priori estimates and topological arguments. In [DM] M. Del Pino and R. Manásevich proved that problem (1) has at least one nontrivial solution when

$$f'_p(0) < \lambda_1(p) < f'_p(\infty). \quad (4)$$

P. Drábek in [Drab] and S. Fucik et al. in [FNSS], focus on the existence of solutions to problem (3) in the case when  $f'_p(\infty)$  is not equal to an eigenvalue of  $-\Delta_p$ . By using topological arguments based on degree theory, they found conditions that allow to show that problem (3) has at least one solution for  $\lambda$  either below  $\lambda_1(p)$  or between  $\lambda_1(p)$  and  $\lambda_2(p)$ . In [DGTU], Drabek et al. study a non-homogeneous version of problem (2) when parameter  $\lambda$  is near  $\lambda_1(p)$ . More recently, Del Pezzo and Quaas in [DQ] generalize the results from [DM] to nonlocal problems involving fractional  $p$ -Laplacian operators. Contrary to conditions in [DM], [DGTU], [Drab], [FNSS], and [DQ], here the  $p$ -derivative at zero and the  $p$ -derivative at infinity are both arbitrarily greater than the first eigenvalue of the  $p$ -Laplace operator.

Regarding quasilinear equations in the radially symmetric case, there has been a lot of research. We mention some works and refer the reader to references therein. For instance, J. Cossio and S. Herrón in [CH] studied problem (1) when  $\Omega$  is the unit ball in  $\mathbb{R}^N$  and the  $p$ -derivative of the nonlinearity at zero is greater than  $\mu_j(p)$ , the  $j$ -radial eigenvalue of the  $p$ -Laplace operator, and the  $p$ -derivative at infinity is equal to the  $p$ -derivative at zero. They showed that problem (1) has  $4j - 1$  radially symmetric solutions. In such a reference, the authors used bifurcation theory and the fact that in the radially symmetric case (1) reduces to an ordinary differential equation. J. Cossio, S. Herrón, and C. Vélez in [CHV2] studied problem (1) in the radially symmetric case, when  $\Omega$  is the unit ball in  $\mathbb{R}^N$  and the problem is  $p$ -superlinear at the origin. They proved that problem (1) has infinitely many solutions. The main tool that they used is the shooting method. M. Del Pino and R. Manásevich in [DM] studied the existence of multiple nontrivial solutions for a quasilinear boundary value problem under radial symmetry; they extended the global bifurcation theorem of P. Rabinowitz (see [R2]) and proved the existence of nontrivial solutions for that kind of problems. In [GS], García-Melián and Sabina de Lis study uniqueness for quasilinear problems in radially symmetric domains.

The paper is organized as follows: In Section 2 we establish some lemmas which will be used to prove Theorem A. We apply a nonlinear version of the strong maximum principle due to J. L. Vázquez (see [V]) to prove that if  $u$  is a weak solution to problem (3) then  $\|u\|_{L^\infty} \neq \alpha$ . We also apply an interpolation theorem due to A. Le (see Theorem 2.1) to show that the function  $(u, \lambda) \mapsto \|u\|_{L^\infty}$  is continuous, where  $(u, \lambda)$  is a solution of (3). In Section 3 we prove Theorem A.

## 2 Preliminary results

Let us recall the definition of weak solutions to problem (3). Given  $\lambda > 0$ , we say a function  $u \in W_0^{1,p}$  solves (3) in the weak sense provided that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} \lambda f(u) v \, dx, \quad \forall v \in W_0^{1,p}. \quad (5)$$

Suppose  $u \in W_0^{1,p}$  solves (3) in the weak sense. Hypothesis  $(f_1)$  implies that

$$\Delta_p u = -\lambda f(u) \leq \lambda C_f |u|^{p-1} \in L_{\text{Loc}}^1(\Omega) \quad (6)$$

and

$$-u \Delta_p u = \lambda u f(u) \leq \lambda C_f |u|^p. \quad (7)$$

From (6) and (7) it follows that  $u \in L^\infty(\Omega)$  (see, for instance, Theorem 6.2.6, p. 737, of [GP]).

The following lemma states a regularity result of any weak solution  $u$  of (3). This result is a particular case of a theorem of Lieberman [Li] (cf. also Di Benedetto [Di]).

**Lemma 1.** *If  $u \in W^{1,p}(\Omega) \cap L^\infty$  and  $\Delta_p u \in L^\infty$  then  $u \in C^{1,\beta}(\overline{\Omega})$  with  $\beta \in (0, 1)$  and*

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C,$$

with  $C > 0$ ; both  $\beta$  and  $C$  depend only on  $N, p, \lambda, \|u\|_{L^\infty}$ , and  $\|\Delta_p u\|_{L^\infty}$ .

The next lemma is a consequence of a nonlinear version of the strong maximum principle due to J. L. Vázquez ([V]).

**Lemma 2.** *If  $u \in C^{1,\beta}(\overline{\Omega})$  is a solution of (3) with  $\lambda > 0$ , then  $\|u\|_{L^\infty} \neq \alpha$ .*

*Proof.* We argue by contradiction. Assume  $u \in C^{1,\beta}(\overline{\Omega})$  is a solution of (3) with  $\lambda > 0$  such that  $\|u\|_{L^\infty} = \alpha$ . Since

$$\lambda |f(u)| \leq \lambda C_f \|u\|_{L^\infty}^{p-1},$$

it follows that

$$-\Delta_p u = \lambda f(u) \in L_{loc}^2(\Omega).$$

We consider the function  $\alpha - u \in C^1(\overline{\Omega})$ ,  $\alpha - u \geq 0$  in  $\Omega$ .

$$\Delta_p(\alpha - u) = \operatorname{div}(|\nabla(\alpha - u)|^{p-2} \nabla(\alpha - u)) = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) \in L_{loc}^2(\Omega).$$

Since  $f(\alpha) \leq 0$ , we see that

$$\begin{aligned} \Delta_p(\alpha - u) &= \lambda f(u) = \lambda f(\alpha - (\alpha - u)) \leq \lambda f(\alpha - (\alpha - u)) - \lambda f(\alpha) \\ &\leq \lambda |f(\alpha - (\alpha - u)) - f(\alpha)| \\ &\leq \lambda C_f |\alpha - u|^{p-1}. \end{aligned} \quad (8)$$

Let us define  $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$  by  $\xi(s) = \lambda C_f s^{p-1}$ . We see that  $\xi$  is continuous, increasing function, such that  $\xi(0) = 0$  and

$$\int_0^1 \frac{1}{(s \xi(s))^{\frac{1}{p}}} ds = c \int_0^1 \frac{1}{(s s^{p-1})^{\frac{1}{p}}} ds = c \ln s \Big|_0^1 = +\infty.$$

Hence, Vázquez maximum principle (see [V]) implies  $\alpha - u > 0$  in  $\Omega$ , i.e.  $u < \alpha$  in  $\Omega$ . Thus  $\|u\|_{L^\infty} < \alpha$ , which contradicts our initial assumption.  $\square$

In the proof of Theorem A inequalities (9) and (10) of the following lemma will play an important role. These inequalities essentially come from the arguments leading to regularity results due to [Di], [T] and [Li].

**Lemma 3.** *There exist positive constants  $K_1 := K_1(|\Omega|, N, C_f, p, \lambda)$  and  $K_2 := K_2(|\Omega|, N, C_f, p, \lambda)$  such that if  $u \in W_0^{1,p}(\Omega)$  is a solution of (3) then*

$$\|u\|_{W_0^{1,p}} \leq K_1 \|u\|_{L^\infty} \quad (9)$$

and

$$\|u\|_{L^\infty} \leq K_2 \|u\|_{W_0^{1,p}}. \quad (10)$$

Moreover,  $K_1$  and  $K_2$  are bounded if  $\lambda$  is bounded.

*Proof.* Let  $u \in W_0^{1,p}(\Omega)$  be a solution of (3). Using the definition of weak solution and hypothesis  $(f_1)$  it follows that

$$\|u\|_{W_0^{1,p}}^p = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla u \, dx = \int_{\Omega} \lambda u f(u) \, dx \leq |\Omega| \lambda C_f \|u\|_{L^\infty}^p. \quad (11)$$

Defining  $K_1 := (|\Omega| \lambda C_f)^{\frac{1}{p}}$ , inequality (9) follows from (11).

Using (6), (7), and a *boot-strap* argument (see, for instance, the proof of Theorem 6.2.6 in [GP]) we get that there exists a constant  $K := K(|\Omega|, N, C_f, p, \lambda) > 0$ , which is bounded when  $\lambda$  is bounded, such that

$$\|u\|_{L^\infty} \leq K \|u\|_{L^{p_0}}, \quad (12)$$

where  $p_0 = \frac{Np}{N-p}$  is a critical Sobolev exponent. Since  $W_0^{1,p}(\Omega)$  is continuously embedded in  $L^{p_0}(\Omega)$ , we see that

$$\|u\|_{L^{p_0}} \leq c_0 \|u\|_{W_0^{1,p}}, \quad (13)$$

for a constant  $c_0 > 0$ . From (12) and (13) we get a constant  $K_2 > 0$  satisfying inequality (10). The proof of Lemma 3 is complete.  $\square$

Let us define

$$S = \{(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : u \neq 0 \text{ and } u = (-\Delta_p)^{-1}(\lambda f(u))\}, \quad (14)$$

where the inverse of the  $p$ -Laplace operator  $L := (-\Delta_p)^{-1} : L^\infty(\Omega) \longrightarrow C^{1,\beta}(\overline{\Omega})$  is known to be a continuous and compact mapping (see [Di] and [Li]). We will make use of the next lemma in the proof of Theorem A.

**Lemma 4.** *The function  $\mathcal{N}_\infty : \overline{S} \subset W_0^{1,p}(\Omega) \times \mathbb{R} \longrightarrow \mathbb{R}$  defined as  $(u, \lambda) \mapsto \|u\|_{L^\infty}$  is continuous.*

*Proof.* We commence by observing that if  $(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R}$  is a limit point of  $S$  then  $u = (-\Delta_p)^{-1}(\lambda f(u))$ , and so  $\|u\|_{L^\infty}$  is well-defined on all  $\overline{S}$ . Let us take  $(u, \lambda_u), (v_n, \lambda_{v_n}) \in \overline{S}$  such that  $(v_n, \lambda_{v_n}) \rightarrow (u, \lambda_u)$ . Let us try to estimate  $|\mathcal{N}_\infty(v_n, \lambda_{v_n}) - \mathcal{N}_\infty(u, \lambda_u)|$ .

$$\begin{aligned} \|v_n - u\|_\infty &= \|L(\lambda_{v_n} f(v_n)) - L(\lambda_u f(u))\|_\infty \\ &= \|\lambda_{v_n}^{\frac{1}{p-1}} L(f(v_n)) - \lambda_u^{\frac{1}{p-1}} L(f(u))\|_\infty \\ &\leq \lambda_{v_n}^{\frac{1}{p-1}} \|L(f(v_n)) - L(f(u))\|_\infty + |\lambda_{v_n}^{\frac{1}{p-1}} - \lambda_u^{\frac{1}{p-1}}| \|L(f(u))\|_\infty. \end{aligned} \quad (15)$$

Let us define

$$u^* = L(f(u)) \text{ and } v_n^* = L(f(v_n)). \quad (16)$$

To estimate  $\|L(f(v_n)) - L(f(u))\|_\infty$  we will need an interpolation type inequality between  $C^{1,0}(\overline{\Omega})$ ,  $C^{1,\beta}(\overline{\Omega})$ , and  $W^{1,p}(\Omega)$ . We will use the following interpolation theorem due to A. Le ([Le]).

**Theorem 2.1.** *There exist constants  $c > 0$  and  $0 < \theta < 1$  such that for any  $u \in C^{1,\beta}(\overline{\Omega}) \cap W^{1,p}(\Omega)$ ,*

$$\|u\|_{1,0} \leq c \|u\|_{C^{1,\beta}}^{1-\theta} \|u\|_{W^{1,p}}^\theta. \quad (17)$$

Since  $u^*, v_n^* \in C^{1,\beta}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ , by using the previous theorem and Poincarè inequality we see that

$$\begin{aligned} \|L(f(v_n)) - L(f(u))\|_\infty &\leq \|L(f(v_n)) - L(f(u))\|_{1,0} \\ &= \|v_n^* - u^*\|_{1,0} \\ &\leq c \|v_n^* - u^*\|_{C^{1,\beta}}^{1-\theta} \|v_n^* - u^*\|_{W_0^{1,p}}^\theta. \end{aligned} \quad (18)$$

We claim that there exists  $C > 0$  such that

$$\|v_n^* - u^*\|_{C^{1,\beta}}^{1-\theta} \leq C. \quad (19)$$

To prove (19) we first show that there exists  $M_1 > 0$  such that  $u, v_n \in B_{M_1}^\infty$ , the ball with radius  $M_1$  in  $L^\infty$ . Since  $v_n \rightarrow u$  in  $W_0^{1,p}$ ,  $\|v_n\|_{W_0^{1,p}}$ ,  $\|v_n\|_{L^{p_0}}$ ,  $\|u\|_{W_0^{1,p}}$ , and  $\|u\|_{L^{p_0}}$  are bounded by a constant. From Lemma 3 we have

$$\|u\|_{L^\infty} \leq K \|u\|_{W_0^{1,p}} \quad \text{and} \quad \|v_n\|_{L^\infty} \leq K \|v_n\|_{W_0^{1,p}}, \quad (20)$$

where  $K$  denotes a positive constant. Thus, there exists  $M_1 > 0$  such that

$$u, v_n \in B_{M_1}^\infty. \quad (21)$$

Combining (21),

$$\|f(u)\|_{L^\infty} \leq C_f \|u\|_{L^\infty}^{p-1}, \quad \text{and} \quad \|f(v_n)\|_{L^\infty} \leq C_f \|v_n\|_{L^\infty}^{p-1}, \quad (22)$$

we see that there exists  $M_2 > 0$  such that

$$\|f(u)\|_{L^\infty} \leq M_2 \quad \text{and} \quad \|f(v_n)\|_{L^\infty} \leq M_2. \quad (23)$$

As we mentioned above, from the regularity results the inverse of the p-Laplace operator

$$L := (-\Delta_p)^{-1} : L^\infty(\Omega) \longrightarrow C^{1,\beta}(\overline{\Omega}) \quad (24)$$

is a continuous and compact mapping. An immediate consequence of (23) and (24) is that there exists  $M > 0$  such that

$$\|u^*\|_{C^{1,\beta}} \leq M \quad \text{and} \quad \|v_n^*\|_{C^{1,\beta}} \leq M. \quad (25)$$

Now (25) implies that there exists  $C > 0$  such that

$$\|v_n^* - u^*\|_{C^{1,\beta}}^{1-\theta} \leq C, \quad (26)$$

which proves (19). From (18), (19), and (26) we see that

$$\|L(f(v_n)) - L(f(u))\|_\infty \leq C \|v_n^* - u^*\|_{W_0^{1,p}}^\theta. \quad (27)$$

Since

$$v_n^* = \frac{v_n}{\lambda_{v_n}^{\frac{1}{p-1}}} \quad \text{and} \quad u^* = \frac{u}{\lambda_u^{\frac{1}{p-1}}} \quad (28)$$

it follows that

$$\begin{aligned} \|L(f(v_n)) - L(f(u))\|_\infty &\leq C \|v_n \lambda_{v_n}^{-\frac{1}{p-1}} - u \lambda_u^{-\frac{1}{p-1}}\|_{W_0^{1,p}}^\theta \\ &= \frac{C}{\lambda_u^{\frac{1}{p-1}} \lambda_{v_n}^{\frac{1}{p-1}}} \|v_n \lambda_u^{\frac{1}{p-1}} - u \lambda_{v_n}^{\frac{1}{p-1}}\|_{W_0^{1,p}}^\theta \\ &= \frac{C}{\lambda_u^{\frac{1}{p-1}} \lambda_{v_n}^{\frac{1}{p-1}}} \|v_n (\lambda_u^{\frac{1}{p-1}} - \lambda_{v_n}^{\frac{1}{p-1}}) + \lambda_{v_n}^{\frac{1}{p-1}} (v_n - u)\|_{W_0^{1,p}}^\theta \\ &\leq \frac{C}{\lambda_u^{\frac{1}{p-1}} \lambda_{v_n}^{\frac{1}{p-1}}} \|v_n\|_{W_0^{1,p}} |\lambda_u^{\frac{1}{p-1}} - \lambda_{v_n}^{\frac{1}{p-1}}| + |\lambda_{v_n}|^{\frac{1}{p-1}} \|v_n - u\|_{W_0^{1,p}}^\theta. \end{aligned} \quad (29)$$

Because the sequences  $\{\|v_n\|_{W_0^{1,p}}\}$  and  $\{\lambda_{v_n}\}$  are bounded, there exists  $C_1$  such that

$$\|L(f(v_n)) - L(f(u))\|_\infty \leq C_1 |\lambda_u^{\frac{1}{p-1}} - \lambda_{v_n}^{\frac{1}{p-1}}| + \|v_n - u\|_{W_0^{1,p}}^\theta. \quad (30)$$

From (15), (30),  $\lambda_{v_n} \rightarrow \lambda_u$ , and  $v_n \rightarrow u$  in  $W_0^{1,p}$  it follows that

$$|\mathcal{N}_\infty(v_n, \lambda_{v_n}) - \mathcal{N}_\infty(u, \lambda_u)| \rightarrow 0, \quad (31)$$

which proves the lemma.  $\square$

### 3 Proof Theorem A

Let  $f$  be a function satisfying the hypotheses  $(f_1) - (f_4)$ . Because of regularity theory (see [Di] and [Li]), the problem of finding solutions  $u \in C^{1,\beta}(\overline{\Omega})$  to (3) is equivalent to find elements  $u \in W_0^{1,p}(\Omega)$  such that

$$u = (-\Delta_p)^{-1}(\lambda f(u)). \quad (32)$$

We will prove that there are nontrivial solutions of (32) when  $\lambda = 1$ , i.e. four nontrivial solutions of (1).

Let  $f^+ : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f^+(t) = f(t)$  for  $t \geq 0$ , and  $f^+(t) = 0$  for  $t < 0$ . Similarly, let  $f^- : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f^-(t) = f(t)$  for  $t \leq 0$ , and  $f^-(t) = 0$  for  $t > 0$ . We observe that  $f$  can be written as

$$f(t) = f'_p(\infty)|t|^{p-2}t + g(t),$$

where  $g(t)/|t|^{p-2}t \rightarrow 0$  as  $|t| \rightarrow \infty$ , and also

$$f(t) = f'_p(0)|t|^{p-2}t + \widehat{g}(t),$$

where  $\widehat{g}(t)/|t|^{p-2}t \rightarrow 0$  as  $t \rightarrow 0$ . From Vázquez maximum principle (see [V]), we have the following lemma.

**Lemma 5.** *If  $u \in W_0^{1,p}(\Omega) \setminus \{0\}$  satisfies  $u = (-\Delta_p)^{-1}(\lambda f^+(u) + \tau)$ , where  $\lambda > 0$  and  $\tau \geq 0$ , then  $u \in C^{1,\beta}(\overline{\Omega})$  for some  $\beta \in (0, 1)$ ,  $u > 0$  on  $\Omega$  and  $\frac{\partial u}{\partial \vec{n}} < 0$  (where  $\vec{n}$  denotes the outer unit normal on  $\partial\Omega$ ).*

**Remark:** taking  $\tau = 0$  in the previous lemma, we observe that if  $u \in W_0^{1,p}(\Omega)$  is a solution of

$$u = (-\Delta_p)^{-1}(\lambda f^+(u)) \quad (33)$$

and  $\lambda > 0$ , then  $u > 0$  on  $\Omega$ . Thus  $u$  satisfies (32), i.e.  $(u, \lambda) \in S$ . In a similar way, if  $u \in W_0^{1,p}(\Omega)$  is a nontrivial solution of

$$u = (-\Delta_p)^{-1}(\lambda f^-(u)) \quad (34)$$

and  $\lambda > 0$ , then  $u < 0$  on  $\Omega$ . Thus  $u$  satisfies (32), i.e.  $(u, \lambda) \in S$ .

We define

$$S^+ = \{(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : u \neq 0 \text{ and } u = (-\Delta_p)^{-1}(\lambda f^+(u))\}$$

and

$$S^- = \{(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : u \neq 0 \text{ and } u = (-\Delta_p)^{-1}(\lambda f^-(u))\}.$$

As we mentioned above, we use bifurcation theory (see [Ra], [R], [R2] and [AM]) to prove Theorem A. Let us recall that, in our framework,  $(0, \lambda^*)$  is a *bifurcation point from zero* for equation  $u = (-\Delta_p)^{-1}(\lambda f^+(u))$  if  $(0, \lambda^*) \in \overline{S^+}$  or, equivalently, if there exists a sequence  $\{(u_n, \lambda_n)\}_n$  in  $S^+$  which converges to  $(0, \lambda^*)$ . Also,  $(\infty, \lambda^*)$  or simply  $\lambda^*$  is a *bifurcation point from infinity* for equation  $u = (-\Delta_p)^{-1}(\lambda f^+(u))$  if there exists a sequence  $\{(u_n, \lambda_n)\}_n$  in  $S^+$  such that  $\lambda_n \rightarrow \lambda^*$  and  $\|u_n\|_{W_0^{1,p}} \rightarrow \infty$  as  $n \rightarrow \infty$ . Similar definitions apply for equation  $u = (-\Delta_p)^{-1}(\lambda f^-(u))$ .

First we present an argument using bifurcation from infinity to show the existence of two one-sign solutions of (1). Secondly, we use bifurcation from zero to show the existence of two additional one-sign solutions. At the end of this section we include a bifurcation diagram which summarizes the arguments presented below.

### 3.1 Bifurcation from infinity

Let us define  $\Psi_+ : W_0^{1,p}(\Omega) \times \mathbb{R} \rightarrow W_0^{1,p}(\Omega)$  by

$$\Psi_+(z, \lambda) = \begin{cases} z - \|z\|^2(-\Delta_p)^{-1} \left[ \lambda f^+ \left( \frac{z}{\|z\|^2} \right) \right] & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

and  $\Psi_-$  in the same way, changing  $f^+$  by  $f^-$ . The following result will be used to prove the existence of two one-sign solutions for problem (1).

**Theorem 3.1.**  *$(\infty, \lambda_1/f'_p(\infty))$  is the unique bifurcation point from infinity for equation (33). More precisely, there exists a connected component  $\Sigma_\infty^+$  of  $S^+$  bifurcating from  $(\infty, \lambda_1/f'_p(\infty))$  which corresponds to an unbounded connected component  $\Gamma_\infty^+$  of*

$$\Gamma^+ = \{(z, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : z \neq 0 \text{ and } \Psi_+(z, \lambda) = 0\},$$

*emanating from the trivial solution of  $\Psi_+(z, \lambda) = 0$  at  $(0, \lambda_1/f'_p(\infty))$ . Analogously, the point  $(\infty, \lambda_1/f'_p(\infty))$  is the unique bifurcation from infinity for equation (34). More precisely, there exists a connected component  $\Sigma_\infty^-$  of  $S^-$  bifurcating from  $(\infty, \lambda_1/f'_p(\infty))$  which corresponds to an unbounded connected component  $\Gamma_\infty^-$  of*

$$\Gamma^- = \{(z, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : z \neq 0 \text{ and } \Psi_-(z, \lambda) = 0\},$$

*emanating from the trivial solution of  $\Psi_-(z, \lambda) = 0$  at  $(0, \lambda_1/f'_p(\infty))$ .*

**Remark:** As we mentioned in the introduction above, Theorem 3.1 is inspired by a corresponding result in the semilinear case due to Ambrosetti and Hess (see [AH] and Section 4.4 in [AM]), and by Theorem 4.1 in [AGP] (see also [DM]). The proof we present below closely follows the ideas from [AH], [AM] and [AGP], but our arguments have several differences with respect to these references. First, as expected, a lot of technicalities arise when trying to adapt the  $\Delta$ -approach from [AH] and [AM] to the  $\Delta_p$  nonlinear operator. Second, our hypotheses on  $f$  slightly differ from those in Theorem 4.1 of [AGP] (ours are a little less restrictive near infinity) and, in the proof presented in [AGP], several details are omitted. And third, our choice of functional spaces is different from both references. For the sake of completeness we include full details here.

In order to prove Theorem 3.1 we need the following lemmas.

**Lemma 6.** *Let  $J \subset \mathbb{R}^+$  be a compact interval such that  $\lambda_\infty := \lambda_1/f'_p(\infty) \notin J$ . Then*



- a) There exists  $r > 0$  such that  $u \neq (-\Delta_p)^{-1}(\lambda f^+(u))$  for every  $\lambda \in J$  and every  $u \in W_0^{1,p}(\Omega)$  with  $\|u\|_{W_0^{1,p}} \geq r$ .
- b)  $(\infty, \lambda_1/f'_p(\infty))$  is the unique bifurcation point from infinity for equation (33).
- c)  $i(\Psi_+(\cdot, \lambda), 0) = 1$  for every  $\lambda < \lambda_\infty$  (here,  $i(\Psi_+(\cdot, \lambda), 0)$  denotes the index of  $\Psi_+(\cdot, \lambda)$  with respect to zero).

*Proof.* In order to prove a) we argue by contradiction. Assume there exist a sequence  $\{\lambda_n\}_n \subset J$  and a sequence  $\{u_n\}_n \subset W_0^{1,p}(\Omega)$  such that  $\|u_n\|_{W_0^{1,p}} \rightarrow +\infty$  and

$$u_n = (-\Delta_p)^{-1}(\lambda_n f^+(u_n)) \quad \text{for every } n \in \mathbb{N}. \quad (35)$$

Because of Lemma 5,  $u_n \geq 0$  for every  $n$ . Dividing (35) by  $\|u_n\|_{W_0^{1,p}}$  we get

$$\frac{u_n}{\|u_n\|_{W_0^{1,p}}} = (-\Delta_p)^{-1} \left( \frac{\lambda_n f'_p(\infty) u_n^{p-1} + \lambda_n g(u_n)}{\|u_n\|_{W_0^{1,p}}^{p-1}} \right) \quad \text{for every } n \in \mathbb{N}, \quad (36)$$

where  $g(t)/|t|^{p-2}t \rightarrow 0$  as  $t \rightarrow \infty$ . What follows is a standard compactness argument. Indeed, since  $\{u_n/\|u_n\|_{W_0^{1,p}}\}_n$  is a bounded sequence in  $W_0^{1,p}(\Omega)$ , there exists a subsequence, for which we keep the same notation,  $\bar{v} \in W_0^{1,p}(\Omega)$  and  $h \in L^p(\Omega)$  such that

$$\left\{ \begin{array}{l} \frac{u_n}{\|u_n\|_{W_0^{1,p}}} \rightharpoonup \bar{v} \quad \text{weakly in } W_0^{1,p}(\Omega) \\ \frac{u_n}{\|u_n\|_{W_0^{1,p}}} \rightarrow \bar{v} \quad \text{strongly in } L^p(\Omega) \\ \frac{u_n(x)}{\|u_n\|_{W_0^{1,p}}} \rightarrow \bar{v}(x) \quad \text{a.e. } x \in \Omega \\ \frac{u_n(x)}{\|u_n\|_{W_0^{1,p}}} \leq h(x) \quad \text{a.e. } x \in \Omega. \end{array} \right. \quad (37)$$

Now, let us verify  $u_n^{p-1}/\|u_n\|^{p-1} \rightharpoonup \bar{v}^{p-1}$  and  $g(u_n)/\|u_n\|^{p-1} \rightharpoonup 0$  weakly in  $L^{p'}(\Omega)$ , where  $1/p + 1/p' = 1$ . Let  $\omega \in L^p(\Omega)$ . Then, from (37),

$$\frac{u_n^{p-1}(x)\omega(x)}{\|u_n\|^{p-1}} \rightarrow \bar{v}^{p-1}(x)\omega(x) \quad \text{a.e. } x \in \Omega \quad \text{and} \quad \frac{u_n^{p-1}\omega}{\|u_n\|^{p-1}} \leq |h|^{p-1}\omega.$$

Since  $h \in L^p(\Omega)$ ,  $|h|^{p-1} \in L^{p'}(\Omega)$ . Hence, dominated convergence theorem implies that

$$\int_{\Omega} \frac{u_n^{p-1}\omega}{\|u_n\|^{p-1}} dx \rightarrow \int_{\Omega} \bar{v}^{p-1}\omega dx \quad \text{as } n \rightarrow \infty.$$

Since this holds true for every  $\omega \in L^p(\Omega)$ , Riesz representation theorem guarantees that  $u_n^{p-1}/\|u_n\|^{p-1} \rightharpoonup \bar{v}^{p-1}$  weakly in  $L^{p'}(\Omega)$ . In order to verify  $g(u_n)/\|u_n\|^{p-1} \rightharpoonup 0$  weakly in  $L^{p'}(\Omega)$ , we take  $\varepsilon > 0$  and then, since  $g(t)/|t|^{p-2}t \rightarrow 0$  as  $t \rightarrow \infty$ , there exists  $M_\varepsilon > 0$  such that

$$t > M_\varepsilon \implies |g(t)| < \varepsilon t^{p-1}. \quad (38)$$

Given  $n \in \mathbb{N}$ , we observe that

$$\int_{\Omega} \frac{g(u_n)}{\|u_n\|^{p-1}} \omega dx = \int_{|u_n| > M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}} \omega dx + \int_{|u_n| \leq M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}} \omega dx. \quad (39)$$

Regarding the first integral on the right-hand side of (39), from (38), Hölder inequality, and the continuity of the embedding, we get

$$\begin{aligned} \left| \int_{|u_n| > M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}} \omega \, dx \right| &= \int_{|u_n| > M_\varepsilon} \frac{|g(u_n)|}{u_n^{p-1}} \frac{u_n^{p-1}}{\|u_n\|^{p-1}} |\omega| \, dx \leq \varepsilon \int_{|u_n| > M_\varepsilon} \frac{u_n^{p-1}}{\|u_n\|^{p-1}} |\omega| \, dx \\ &\leq \varepsilon \|\omega\|_{L^p} \left\| \frac{u_n^{p-1}}{\|u_n\|^{p-1}} \right\|_{L^{p'}} \leq C\varepsilon \|\omega\|_{L^p}. \end{aligned} \quad (40)$$

With respect to the second integral on the right-hand side of (39), we have

$$\left| \int_{|u_n| \leq M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}} \omega \, dx \right| = \|g\|_{L^\infty[0, M_\varepsilon]} \int_{|u_n| \leq M_\varepsilon} \frac{|\omega|}{\|u_n\|^{p-1}} \, dx \leq \frac{\|g\|_{L^\infty[0, M_\varepsilon]}}{\|u_n\|^{p-1}} \|\omega\|_{L^1}. \quad (41)$$

Since  $\varepsilon > 0$  is fixed,  $\|g\|_{L^\infty[0, M_\varepsilon]}$  is fixed. The right-hand side of (41) tends to zero as  $n \rightarrow \infty$ , because  $\|u_n\|_{W_0^{1,p}} \rightarrow +\infty$ . Thus, from (39), (40), (41), and the fact that  $\omega \in L^p(\Omega)$  is arbitrary, we conclude  $g(u_n)/\|u_n\|^{p-1} \rightharpoonup 0$  weakly in  $L^{p'}(\Omega)$ .

We then have that the argument on the right-hand side in (36) converges weakly to  $\bar{\lambda} f'_p(\infty) \bar{v}$  in  $L^{p'}(\Omega)$ , for some  $\bar{\lambda} \in J$ . As  $(-\Delta_p)^{-1} : L^{p'}(\Omega) \rightarrow W_0^{1,p}(\bar{\Omega})$  is compact, from (36) we get a further subsequence  $\{\frac{u_n}{\|u_n\|_{W_0^{1,p}}}\}_n$  such that

$$\frac{u_n}{\|u_n\|_{W_0^{1,p}}} = (-\Delta_p)^{-1} \left( \frac{\lambda_n f'_p(\infty) u_n^{p-1} + \lambda_n g(u_n)}{\|u_n\|_{W_0^{1,p}}^{p-1}} \right) \rightarrow (-\Delta_p)^{-1} (\bar{\lambda} f'_p(\infty) \bar{v}) \quad \text{as } n \rightarrow \infty \quad (42)$$

strongly  $W_0^{1,p}(\bar{\Omega})$ . From (37) and (42) we conclude

$$(-\Delta_p)^{-1} (\bar{\lambda} f'_p(\infty) \bar{v}) = \bar{v}. \quad (43)$$

We claim  $\bar{v} \neq 0$ . Let us denote  $v_n := u_n/\|u_n\|_{W_0^{1,p}}$  for each  $n$ . From (36), we have that

$$-\Delta_p v_n = \lambda_n f'_p(\infty) v_n^{p-1} + \frac{\lambda_n g(u_n)}{\|u_n\|_{W_0^{1,p}}^{p-1}} \quad \text{for every } n \in \mathbb{N}, \quad (44)$$

in the weak sense. Multiplying (44) by  $v_n$  and integrating we get

$$1 = \|v_n\|_{W_0^{1,p}}^p = \lambda_n f'_p(\infty) \|v_n\|_{L^p}^p + \int_{\Omega} \frac{\lambda_n g(u_n)}{\|u_n\|_{W_0^{1,p}}^{p-1}} v_n \, dx \quad \text{for every } n \in \mathbb{N}. \quad (45)$$

By virtue of (37) and the compactness of  $J \subset \mathbb{R}^+$ , the first term on the right-hand-side in (45) tends to  $\bar{\lambda} f'_p(\infty) \|\bar{v}\|_{L^p}^p$ . Arguing as above, when we proved  $g(u_n)/\|u_n\|^{p-1} \rightharpoonup 0$  weakly in  $L^{p'}(\Omega)$ , one can show

$$\int_{\Omega} \frac{\lambda_n g(u_n)}{\|u_n\|_{W_0^{1,p}}^{p-1}} v_n \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

(double checking (40) and (41), replacing  $\omega$  by  $v_n$ , one can observe that the same argument can be carried out provided that sequence  $\{\|v_n\|_{L^p}^p\}_n$  be bounded, which is our case). Thus, taking limit in (45), we conclude  $1 = \bar{\lambda} f'_p(\infty) \|\bar{v}\|_{L^p}^p$  and so  $\bar{v} \neq 0$ .

Therefore (43) means  $\bar{\lambda}f'_p(\infty)$  is an eigenvalue of  $-\Delta_p$  and  $\bar{v}$  is an associated eigenfunction. This is absurd since  $\bar{v} \geq 0$  (from (37)) and  $\bar{\lambda}f'_p(\infty) \neq \lambda_1$  (since  $\bar{\lambda} \in J$  and  $\lambda_1/f'_p(\infty) \notin J$ ). This contradiction completes our proof of a).

To prove b), again we argue by contradiction. Assume there is a bifurcation point  $\bar{\lambda}$  from  $\infty$  such that  $\bar{\lambda} \neq \lambda_\infty$ . Let  $J \subset \mathbb{R}^+$  be a compact interval such that  $\bar{\lambda} \in J$  and  $\lambda_\infty \notin J$ . Then, there exists a sequence  $\{(u_n, \lambda_n)\} \subset S^+$  such that  $\|u_n\|_{W_0^{1,p}} \rightarrow +\infty$  and  $\lambda_n \in J$  for large  $n \in \mathbb{N}$ . But this contradicts a).

We now prove c). Let  $\lambda < \lambda_\infty$ . Consider  $J = [0, \lambda]$ . For every  $t \in [0, 1]$  we have  $t\lambda \in J$ . From a) it follows that

$$u - (-\Delta_p)^{-1}(t\lambda f^+(u)) \neq 0$$

for every  $\lambda \in J$  and every  $u \in W_0^{1,p}(\Omega)$  with  $\|u\|_{W_0^{1,p}} \geq r$ . For such an  $u$ , taking  $z = u/\|u\|_{W_0^{1,p}}^2$ , we get

$$z - \|z\|^2(-\Delta_p)^{-1}(t\lambda f^+(z/\|z\|^2)) \neq 0$$

for every  $z \in W_0^{1,p}(\Omega)$  such that  $\|z\|_{W_0^{1,p}} \leq 1/r$ . Hence,  $\Psi_+(z, t\lambda) \neq 0$  for every  $z \in W_0^{1,p}(\Omega)$  such that  $0 < \|z\|_{W_0^{1,p}} \leq 1/r$ . Let us define the homotopy  $H : [0, 1] \times W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)$  by  $H(t, u) = \Psi_+(u, t\lambda)$ . Using Leray-Schauder degree invariance under homotopies, we get

$$\deg(H(1, \cdot), B_{1/r}(0), 0) = \deg(H(0, \cdot), B_{1/r}(0), 0)$$

equivalently

$$\deg(\Psi_+(\cdot, \lambda), B_{1/r}(0), 0) = \deg(I, B_{1/r}(0), 0) = 1.$$

□

**Lemma 7.** *The following assertions hold true:*

a) *Let  $\lambda > \lambda_\infty := \lambda_1/f'_p(\infty)$ . Then there exists  $R > 0$  such that  $u \neq (-\Delta_p)^{-1}(\lambda f^+(u) + \tau)$  for every  $\tau \geq 0$  and every positive  $u \in W_0^{1,p}(\Omega)$  such that  $\|u\|_{W_0^{1,p}} \geq R$ .*

b)  *$i(\Psi_+(\cdot, \lambda), 0) = 0$  for all  $\lambda > \lambda_\infty$ .*

*Proof.* In order to prove a) we argue by contradiction. Actually, our argument is similar to the one we used above when proving Lemma 6 part a), but in this case it is more involved because of the  $\tau$ -term. Assume there exist  $\{\tau_n\}_n \subset [0, \infty)$  and a sequence  $\{u_n\}_n \subset W_0^{1,p}(\Omega)$  of nonnegative functions such that  $\|u_n\|_{W_0^{1,p}} \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$u_n = (-\Delta_p)^{-1}(\lambda f^+(u_n) + \tau_n) \quad \text{for every } n \in \mathbb{N}. \quad (46)$$

Since  $f^+(t) = f'_p(\infty)|t|^{p-2}t + g(t)$ , where  $g(t)/|t|^{p-2}t \rightarrow 0$  as  $t \rightarrow +\infty$ , (46) can be written as

$$u_n = (-\Delta_p)^{-1}(\lambda f'_p(\infty)u_n^{p-1} + \lambda g(u_n) + \tau_n) \quad \text{for every } n \in \mathbb{N}. \quad (47)$$

Let  $v_n = u_n/\|u_n\|_{W_0^{1,p}}$  for every  $n \in \mathbb{N}$ . Then  $v_n$  satisfies equation

$$v_n = (-\Delta_p)^{-1} \left( \lambda f'_p(\infty)v_n^{p-1} + \lambda \frac{g(u_n)}{\|u_n\|^{p-1}} + \frac{\tau_n}{\|u_n\|^{p-1}} \right) \quad \text{for all } n \in \mathbb{N}. \quad (48)$$

We may assume (by passing to a subsequence) that either

- i)  $\frac{\tau_n}{\|u_n\|^{p-1}} \rightarrow c \geq 0$  as  $n \rightarrow \infty$ , or  
 ii)  $\frac{\tau_n}{\|u_n\|^{p-1}} \rightarrow +\infty$  as  $n \rightarrow \infty$ .

Let us consider case i). Assume first that  $c = 0$ . Since  $\|v_n\|_{W_0^{1,p}} = 1$  for every  $n \in \mathbb{N}$ , we can suppose (by taking a subsequence) that there exists  $\bar{v} \in W_0^{1,p}(\Omega)$  such that  $v_n \rightharpoonup v$  (weakly) in  $W_0^{1,p}(\Omega)$  and (37) holds true. Arguing as in the proof of Lemma 6,

$$\lambda f'_p(\infty) v_n^{p-1} \rightharpoonup \lambda f'_p(\infty) v^{p-1} \quad \text{and} \quad \frac{g(u_n)}{\|u_n\|^{p-1}} \rightharpoonup 0 \quad \text{weakly in} \quad L^{p'}(\Omega), \quad (49)$$

and by our assumption that  $c = 0$  in i),

$$\frac{\tau_n}{\|u_n\|^{p-1}} \rightharpoonup 0 \quad \text{weakly in} \quad L^{p'}(\Omega). \quad (50)$$

Since  $(-\Delta_p)^{-1} : L^{p'}(\Omega) \rightarrow W_0^{1,p}(\Omega)$  is a compact operator, it follows from (48), (49) and (50)

$$v = (-\Delta_p)^{-1} (\lambda f'_p(\infty) v^{p-1}) \Leftrightarrow -\Delta_p v = \lambda f'_p(\infty) v^{p-1}. \quad (51)$$

Arguing as we did above after getting (43), we get that the nonnegative function  $\bar{v}$  is also nonzero. Thus, (51) provides a contradiction since  $\lambda > \lambda_1(p)/f'_p(\infty)$ .

We now assume  $\frac{\tau_n}{\|u_n\|^{p-1}} \rightarrow c > 0$  as  $n \rightarrow \infty$ . Let  $\varepsilon \in (0, \lambda f'_p(\infty) - \lambda_1)$ . Using a standard argument, the following claim can be demonstrated.

**Claim:** there exists a large  $n$  such that  $v_n$  is a weak positive supersolution  $\omega \in W_0^{1,p}(\Omega)$  of problem

$$\begin{cases} -\Delta_p \omega = (\lambda_1 + \varepsilon) \omega^{p-1} & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases} \quad (52)$$

Now, for every  $t > 0$  and a positive eigenfunction  $\phi_1$  corresponding to  $\lambda_1$ ,  $t\phi_1$  is a subsolution of problem (52). Let  $v_n$  be a positive supersolution of (52). Using that  $\frac{\partial v_n}{\partial \vec{n}} < 0$  and  $\frac{\partial \phi_1}{\partial \vec{n}} < 0$  on  $\partial\Omega$  (where  $\vec{n}$  denotes the outer unit normal on  $\partial\Omega$ ), one can prove there exists  $t > 0$  such that  $t\phi_1 \leq v_n$  on  $\Omega$ . Using standard truncation and penalization techniques (see e.g. [DKT], the appendix in [GS], or Section 4.5 in [GP]), it can be proved the existence of a solution  $\omega \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , of problem (52), such that  $t\phi_1 \leq \omega \leq v_n$  in  $\Omega$ . Thus  $\omega$  is a positive eigenfunction corresponding to the eigenvalue  $\lambda_1 + \varepsilon \neq \lambda_1$ . This is a contradiction that shows case i) above cannot actually occur.

Let us now consider case ii). Arguing as in case i), from (48) it follows that, for  $n \in \mathbb{N}$  sufficiently large, inequality  $-\Delta_p v_n \geq \lambda \gamma v_n^{p-1}$  holds true. Then, the same argument as presented in case i) follows, and we also get a contradiction. We have completed the proof of part a).

We now prove b). Let a)  $\lambda > \lambda_\infty$ . From a), taking  $\tau = t$ , we know that  $u \neq (-\Delta_p)^{-1}(\lambda f^+(u) + t)$  for every  $t \in [0, 1]$  and every  $u \in W_0^{1,p}(\Omega)$  with  $\|u\|_{W_0^{1,p}} \geq R$ . Using again inversion  $z = u/\|u\|_{W_0^{1,p}}^2$  and the homogeneity of  $(-\Delta_p)^{-1}$ , we observe that

$$z \neq (-\Delta_p)^{-1} \left( \lambda \|z\|^{2(p-1)} f^+(z/\|z\|^2) + t \right) \quad (53)$$

for every  $t \in [0, 1]$  and every  $z \in W_0^{1,p}(\Omega)$  such that  $0 < \|z\|_{W_0^{1,p}} \leq 1/R$ . Let  $\varepsilon \in (0, 1/R)$ . We now define homotopy  $H : [0, 1] \times B_\varepsilon(0) \longrightarrow W_0^{1,p}(\Omega)$  as

$$H(t, z) = z - (-\Delta_p)^{-1} \left( \lambda \|z\|^{2(p-1)} f^+ (z/\|z\|^2) + t \right) \quad \text{for every } z \neq 0,$$

and  $H(t, 0) := -(-\Delta_p)^{-1}(t)$ . Using the same arguments we used above it can be proved that  $H$  is actually continuous, and also that it is of the form identity – compact.

Using the homotopy invariance property of Leray-Schauder degree, we obtain

$$\deg(H(0, \cdot), B_\varepsilon(0), 0) = \deg(H(1, \cdot), B_\varepsilon(0), 0).$$

On the other hand,  $\deg(H(0, \cdot), B_\varepsilon(0), 0) = \deg(\Psi_+(\cdot, \lambda), B_\varepsilon(0), 0)$  and, from (53) and the definition of  $H$ ,

$$\deg(H(1, \cdot), B_\varepsilon(0), 0) = 0.$$

□

*Proof of Theorem 3.1.* Lemmas 6 and 7 assert that  $i(\Psi_+(\cdot, \lambda), 0) = 1$  when  $\lambda < \lambda_\infty$ , and  $i(\Psi_+(\cdot, \lambda), 0) = 0$  when  $\lambda > \lambda_\infty$ . The fact that these two local degrees are different allows one to repeat the original arguments used by P. Rabinowitz to prove his global bifurcation theorem (see [Ra], [R], and [AM] Sections 4.3 and 4.4). □

We now prove the existence of two solutions for problem (1). Since  $\Sigma_\infty^+$  bifurcates from  $(\infty, \lambda_1/f_p'(\infty))$ , there exist elements  $(u, \lambda) \in \Sigma_\infty^+$  such that  $\|u\|_{W_0^{1,p}(\Omega)}$  is arbitrarily large and  $\lambda$  is near  $\lambda_1/f_p'(\infty)$ . Hence, because of inequality (9) in Lemma 3, there exist elements  $(u, \lambda) \in \Sigma_\infty^+$  such that  $\mathcal{N}_\infty(u, \lambda) = \|u\|_{L^\infty(\Omega)} > \alpha$ . Lemma 4 implies that  $\mathcal{N}_\infty(\overline{\Sigma_\infty^+})$  is connected. Thus, Lemma 2 implies that

$$\|u\|_{L^\infty(\Omega)} > \alpha \quad \forall (u, \lambda) \in \overline{\Sigma_\infty^+}. \quad (54)$$

Because of inequality (10) in Lemma 3,

$$\|u\|_{W_0^{1,p}(\Omega)} > (K_2)^{-1}\alpha \quad \forall (u, \lambda) \in \overline{\Sigma_\infty^+} \cap (W_0^{1,p}(\Omega) \times [0, 2]). \quad (55)$$

Now we claim that there exists an element of the form  $(u_1, 1) \in \overline{\Sigma_\infty^+}$ . Let us argue by contradiction. Assume this is not true. Consider the cylinder

$$P = \{(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : \lambda \in [0, 1], \|u\|_{W_0^{1,p}} \geq (K_2)^{-1}\alpha\}.$$

Hypothesis  $(f_2)$  implies that  $\lambda_1/f_p'(\infty) < 1$ . Therefore, from Theorem 3.1 it follows that  $\text{int}P \cap \overline{\Sigma_\infty^+} \neq \emptyset$ . Also, since  $\Sigma_\infty^+$  corresponds to the unbounded connected component  $\Gamma_\infty^+$  of  $\Gamma^+$ , then  $\text{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P) \cap \overline{\Sigma_\infty^+} \neq \emptyset$ . From (55) and our assumption,  $\partial P \cap \overline{\Sigma_\infty^+} = \emptyset$ . Thus,  $\partial P$  separates  $\overline{\Sigma_\infty^+}$ , i.e.

$$\overline{\Sigma_\infty^+} \subset \text{int}P \cup \text{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P),$$

which contradicts the connectedness of  $\overline{\Sigma_\infty^+}$ . This contradiction shows there exists  $(u_1, 1) \in \overline{\Sigma_\infty^+}$ . From Theorem 3.1,  $u_1 \neq 0$ , i.e.  $(u_1, 1) \in \Sigma_\infty^+ \subset S^+$ . As mentioned above, this means  $u_1 > 0$  on  $\Omega$  and  $u_1$  satisfies (1). In a similar fashion we obtain a negative solution  $v_1$ . The previous argument shows these two solutions have  $L^\infty$ -norm greater than  $\alpha$ .

### 3.2 Bifurcation from zero

First we state the following analogue of Theorem 3.1.

**Theorem 3.2.** *There exists an unbounded connected component  $\Sigma_0^+$  of  $S^+$  so that  $(0, \lambda_1/f_p'(0))$  belongs to  $\overline{\Sigma_0^+}$  and if  $(0, \lambda) \in \overline{\Sigma_0^+}$  then  $\lambda = \lambda_1/f_p'(0)$ . Also, there exists an unbounded connected component  $\Sigma_0^-$  of  $S^-$  such that  $(0, \lambda_1/f_p'(0)) \in \overline{\Sigma_0^-}$  and if  $(0, \lambda) \in \overline{\Sigma_0^-}$  then  $\lambda = \lambda_1/f_p'(0)$ .*

**Remark:** This result is essentially an adaptation of Lemma 3.1 in [DM] to our case, and it can be proved either by following the arguments of [DM] (Theorem 1.1 and Lemma 3.1) or by using the same ideas we used above to prove Theorem 3.1.

We now prove the existence of two additional solutions for problem (1). Since  $(0, \lambda_1/f_p'(0)) \in \overline{\Sigma_0^+}$ , there exist elements  $(u, \lambda) \in \Sigma_0^+$  such that  $\|u\|_{W_0^{1,p}(\Omega)}$  is close to zero and  $\lambda$  is near  $\lambda_1/f_p'(0)$ . Hence, because of inequality (10) in Lemma 3, there exist elements  $(u, \lambda) \in \Sigma_0^+$  such that  $\mathcal{N}_\infty(u, \lambda) = \|u\|_{L^\infty(\Omega)} < \alpha$ . From Lemma 4 it follows that  $\mathcal{N}_\infty(\overline{\Sigma_0^+})$  is connected. Thus, Lemma 2 implies that

$$\|u\|_{L^\infty(\Omega)} < \alpha \quad \forall (u, \lambda) \in \overline{\Sigma_0^+}. \quad (56)$$

Because of inequality (9) in Lemma 3,

$$\|u\|_{W_0^{1,p}(\Omega)} < K_1\alpha \quad \forall (u, \lambda) \in \overline{\Sigma_0^+} \cap (W_0^{1,p}(\Omega) \times [0, 2]). \quad (57)$$

Now we claim that there exists  $(u_2, 1) \in \overline{\Sigma_0^+}$ . Let us argue by contradiction. Assume this is not true. Define the cylinder

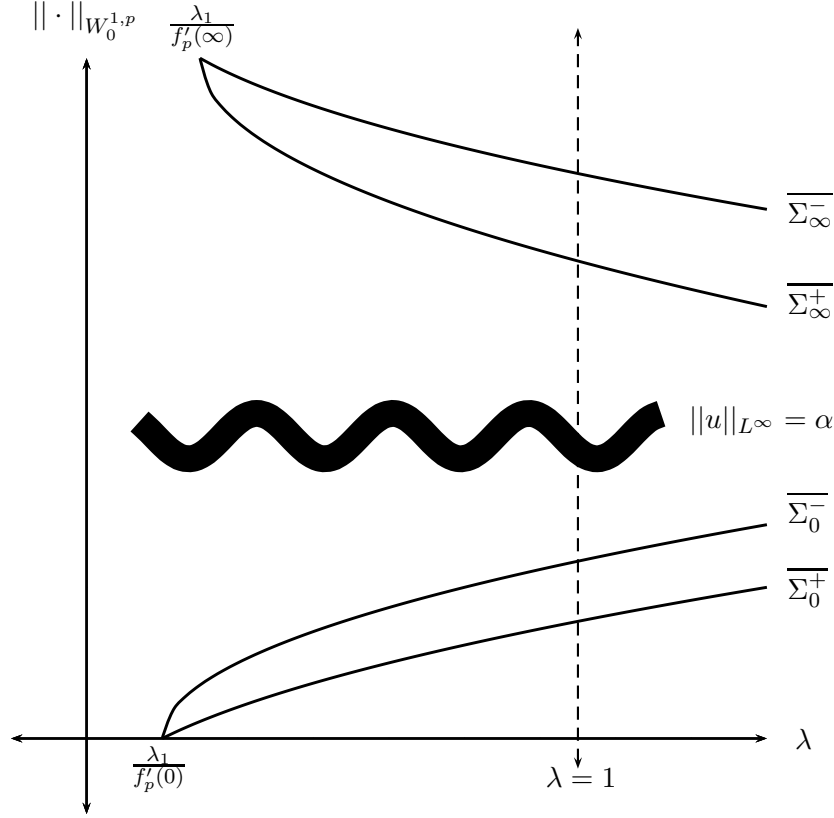
$$P = \{(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : \lambda \in [0, 1], \|u\|_{W_0^{1,p}} \leq K_1\alpha\}.$$

Hypothesis  $(f_2)$  implies that  $\lambda_1/f_p'(0) < 1$ . Therefore, from Theorem 3.2 it follows that  $\text{int}P \cap \overline{\Sigma_0^+} \neq \emptyset$ . Also, the unboundedness of  $\overline{\Sigma_0^+}$  implies  $\text{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P) \cap \overline{\Sigma_0^+} \neq \emptyset$ . From (57) and our assumption,  $\partial P \cap \overline{\Sigma_0^+} = \emptyset$ . Thus,  $\partial P$  separates  $\overline{\Sigma_0^+}$ , i.e.

$$\overline{\Sigma_0^+} \subset \text{int}P \cup \text{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P),$$

which contradicts the connectedness of  $\overline{\Sigma_0^+}$ . This contradiction shows there exists  $(u_2, 1) \in \overline{\Sigma_0^+}$ . From Theorem 3.2,  $u_2 \neq 0$ , i.e.  $(u_2, 1) \in \Sigma_0^+ \subset S^+$ . As mentioned above, this means  $u_2 > 0$  on  $\Omega$  and  $u_2$  satisfies (1).

Arguing in a similar fashion with  $\overline{\Sigma_0^-}$ , the existence of a negative solution  $v_2$  of (1) is obtained. From (56) (and its analogue for  $\overline{\Sigma_0^-}$ ) we have  $\|u_2\|_{L^\infty}, \|v_2\|_{L^\infty} < \alpha$ . We summarize the arguments presented above in the following bifurcation diagram.



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